"In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institution shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, if such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

"

FLOW ANALYSIS OF DISTRIBUTION SYSTEMS CONTAINING ELEVATED RESERVOIRS BY THE HARDY CROSS METHOD

A THESIS

Presented to

the Faculty of the Graduate Division

Georgia Institute of Technology

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Civil Engineering

By

Alison Templeton Adams

December 1955

5 10
12R

FLOW ANALYSIS OF DISTRIBUTION

SYSTEMS CONTAINING ELEVATED RESERVOIRS

BY THE HARDY CROSS METHOD

Approved:

Date Approved by Chairman: Dec. 21, 1955

ACKNOWLEDGMENTS

I wish to express my sincere appreciation to Doctor M. R. Carstens for his suggestion of the problem for this study and for his guidance during its analysis and to Professors C. E. Kindsvater and W. R. Metcalfe for their help in preparing the final manuscript.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SUMMARY

The purpose of the thesis is to analyze, by means of the Hardy Cross method of successive approximations, the distribution of flow throughout a pipe network which contains elevated reservoirs. The methods of analyzing the distribution of flow in a pipe network are surveyed. Derivations of the Hardy Cross method for determining the distribution of flow in pipe networks both with and without elevated reservoirs are presented. A formula for determining the exact value of the Hardy Cross flow correction for any single circuit is derived using the Darcy-Weisbach equation. The rapidity of convergence of the Hardy Cross method is discussed. Analyses of systems which contain elevated reservoirs, while the reservoirs are emptying, are presented for cases of constant flow being taken from the system and of free flow being allowed from the system.

The conclusions that were reached are:

1. The Hardy Cross method can be used to determine the distribution of flow in a system which contains elevated reservoirs. However, in complicated systems, the use of the electric analogy is advisable if the distribution of flow under many different conditions is required.
2. In any system the Hardy Cross flow corrections are usually too large.

3. In any system the convergence of the Hardy Cross method is more rapid when the factors of proportionality representing the resistance of the individual pipes in the system are of relatively the same magnitude.

4. In the solution of a problem more rapid convergence can be achieved if an extra circuit is considered such that each pipe will have two flow corrections to be applied. If any flow correction is large then an average value should be applied. If all flow corrections are small then both should be applied.

5. When a constant flow is taken from a system containing elevated prismatic reservoirs while the reservoirs are emptying, the individual flows in the system and the differences in elevation of the water levels in the reservoirs will eventually reach a steady state in which the flows from the reservoirs are proportional to the areas of the reservoirs providing the system is undisturbed and none of the reservoirs are emptied before this steady state is reached.

CHAPTER I

INTRODUCTION

The purpose of this thesis is to analyze by means of the Hardy Cross method of successive approximations the distribution of flow throughout a pipe network which contains elevated reservoirs. The methods of analyzing the distribution of flow in a pipe network are surveyed. Analyses of systems with reservoirs are presented for cases of constant flow being taken from the system and of free flow being allowed from the system. Physical interpretations of these analyses are presented. The rapidity of convergence of the Hardy Cross method is discussed.

A pipe network consists of the interconnected pipes, valves, elbows, bends and reservoirs which form the water distribution system. The problem is to compute the flows and head losses which occur in the constituent pipes of the system. The various types of problems that are encountered can be divided into three general classifications which depend on the information that is known at the outset of the problem: the flows into and out of the network; the heads at points of flow into and out of the network; and, some combination of the heads and flows.

There are two basic principles which govern the distribution of flow in a pipe network. These principles can readily be seen by examining Fig. 1.

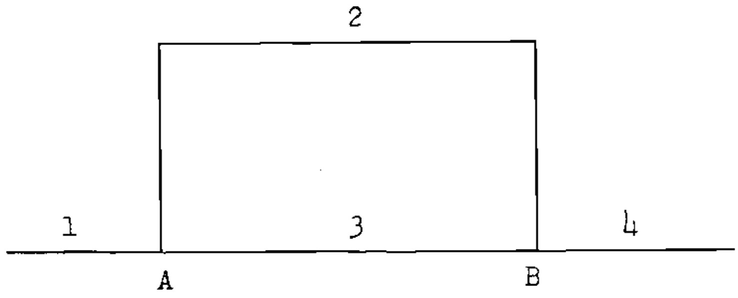

Fig. 1. A Simple Pipe Network

The quantity of water flowing through pipe "1" into junction "A" must obviously equal the quantity of water flowing out of junction "A" through pipes "2" and "3." If positive and negative signs are arbitrarily assigned to flows into and out of a junction, respectively, the first principle can be stated: the algebraic sum of the discharges toward any junction is zero. The water at junction "A" will have some definite value of total head. The water at junction "B" will have some other value of total head which will differ from the value of total head at junction "A" by the head loss which occurs between junctions "A" and "B." Thus, regardless of the route the water takes to get from junction "A" to junction "B," the head loss that occurs must be equal to the difference in total head between junctions "A" and "B." If positive and negative signs are arbitrarily assigned to clockwise and counterclockwise flows and head losses, respectively, the second principle can be stated: the algebraic sum of the head losses around any closed circuit is zero.

In order to effect a solution of the problem the head loss must be correlated with the discharge. The rational Darcy-Weisbach equation or the empirical Hazen-Williams equation are the two most commonly used expressions relating the head loss and the discharge. In the Darcy-Weisbach equation the head loss varies as the second power of the discharge, and in the Hazen-Williams equation the head loss varies as the 1.85 power of the discharge.

CHAPTER II

METHODS OF DETERMINING FLOW DISTRIBUTION IN A PIPE NETWORK

The distribution of flow in a pipe network can be determined by three methods: direct measurement, numerical analysis, or analogy.

Direct Measurements.--Direct measurements are the most accurate means of determining the distribution of flow in a pipe network. However, they can only be used when the network is already in existence and they are, therefore, useless for design purposes. In this method pitot tubes are connected to the various pipes in the system and the flow is determined from the readings of the pitot tubes. If recording devices are utilized, the readings from the pitot tubes can be permanently recorded as a function of time. In this manner the simultaneous discharges of many pipes can be determined.

Numerical Analyses.--The analytical methods for determining the distribution of flow in a pipe network are all based on the trial-and-error principle. They are divided into two groups: the controlled and uncontrolled methods. In the controlled methods the adjustments to the system are based upon a standardized computation procedure such that the adjustments become successively smaller with each progressive step in the problem. In the uncontrolled methods the adjustments are made arbitrarily.

The uncontrolled trial-and-error method has many variations.

The method consists primarily of making some assumptions for the distribution of flow within the network and then calculating the head losses which were caused by these assumed flows, and, finally, determining if the head losses are properly balanced. This process is repeated until the desired accuracy is obtained.

The Freeman graphical method (1) was introduced in 1892. Expansions of this method have been made by Howland (2) in 1934 and Aldrich (3) in 1938. These solutions are obtained by plotting the discharge as an abscissa against the head loss as an ordinate on separate sections of transparent graph paper for each section of pipe. The curves are then overlayed and combined in such a way as to form a master curve which is representative of the system of pipes being considered. Elevated storage tanks and flows into or out of the system at many points can be incorporated in this method. The graphical methods have the advantage of presenting a visual analysis of the flow distribution in a pipe network for varied flows. The disadvantages of these methods are that they are very complicated for an involved network and that they are uncontrolled.

The Hardy Cross method (4) which was introduced in 1936, is a controlled analytical method. It is derived and explained in Chapter III.

Analogies.--There are two analogies which can be used to find the flow distribution in a pipe network: the hydraulic analogy and the electric analogy. The principal disadvantages are the difficulty and expense of constructing the models. The principal advantages are that changes in the network elements and relocations of the demand flows are easily accomplished once the models have been completed.

The possibility of using small hydraulic models (5) to determine the distribution of flow in a pipe network has long been considered. The principal difficulty of the hydraulic model is to represent the prototype pipe with a model having the same hydraulic properties. Present hydraulic models use orifices inserted in pipes or rubber tubing, or constrictions in rubber tubing to represent the pipe in the system. Once the model is constructed the flows in the prototype can be determined by measuring the flows in the model.

Work was begun upon the electric analyzer method (6, 7, 8) in 1933. It is based upon the similarity of the flow of electric current to the flow of water. Three laws govern the flow distribution of water in a network:

1. The algebraic sum of the discharge toward any junction point is zero.
2. The algebraic sum of the head losses around any closed circuit is zero.
3. The head loss is directly proportional to some power of the discharge.

Kirchoff's laws for electric circuits are analogous to the first two laws stated above.

1. The algebraic sum of the current flowing toward any junction point is zero.
2. The algebraic sum of the voltage drops around any closed circuit is zero.

Ohm's law states that the voltage drop is directly proportional to the first power of the current times the resistance. If the current is considered as representing the discharge and the voltage drop is considered as representing the head loss, the analogy between the flow in the electric network and the hydraulic network is inexact only because the voltage drop varies with the first power of the current while the head loss varies with some higher power of the discharge. Non-linear electrical resistors have been developed which make it possible in Ohm's law to cause the voltage drop to vary with any specified power of the current. Thus, an electric network with non-linear resistors can be constructed and used as a model for any hydraulic network. The flows and head losses in the hydraulic network can be determined by measuring the currents and voltage drops in the electric network.

CHAPTER III

THE HARDY CROSS METHOD OF DETERMINING FLOW DISTRIBUTION IN A PIPE NETWORK

Derivation of the Hardy Cross method.--The Hardy Cross method (4) of determining the flow distribution in a pipe network, which was introduced in 1936, is a controlled analytical method. It can be derived from the basic principles which govern flow in a pipe network.

1. The algebraic sum of the discharge toward any junction point is zero.
2. The algebraic sum of the head losses around any closed circuit is zero.
3. The head loss varies with some power of the discharge.

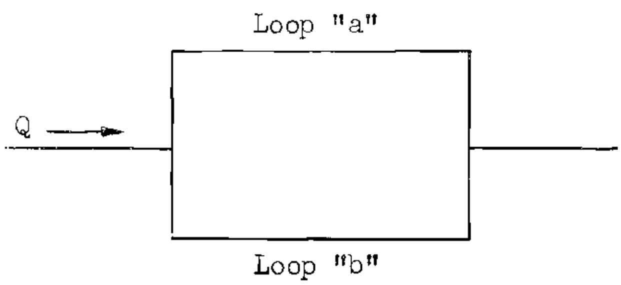

Fig. 2. A Simple Pipe Network

In Fig. 2, when the flow is correctly distributed, the head loss around loop "a," h_a, minus the head loss around loop "b," h_b, is zero, or,

$$h_a - h_b = 0$$

The total flow through the network is Q. The flows through loop "a" and loop "b," respectively, are unknown but are given assumed values Q_a and Q_b such that the sum of Q_a and Q_b equals the total flow, Q. These assumed values, Q_a and Q_b, differ from the correct values of flow by some unknown quantity of flow ΔQ. If the assumed value of flow Q_a is converted to the correct value of flow by the addition of ΔQ, then the assumed value of flow, Q_b, must be converted to the correct value of flow by the subtraction of ΔQ. Thus,

$$Q_a + Q_b = Q$$

and

$$(Q_a + \Delta Q) + (Q_b - \Delta Q) = Q$$

Since the head loss, h, varies with some power, n, of the discharge, Q, or $h = kQ^n$ in which k is a constant of proportionality,

$$h_a = k_a (Q_a + \Delta Q)^n$$

$$h_b = k_b (Q_b - \Delta Q)^n$$

Since if the flow is properly distributed $h_a - h_b = 0$,

$$k_a (Q_a + \Delta Q)^n - k_b (Q_b - \Delta Q)^n = 0$$

If the quantities raised to the power n, are expanded by the binominal theorem and powers of ΔQ greater than unity are neglected,

$$k_a\ (Q_a^n + n\ Q_a^{n-1}\ \Delta Q \ldots) - k_b\ (Q_b - n\ Q_b^{n-1}\ \Delta Q \ldots) = 0$$

Solving for ΔQ,

$$\Delta Q = - \frac{k_a Q_a^n - k_b Q_b^n}{n\ k_a Q_a^{n-1} + n\ k_b Q_b^{n-1}}$$

The general equation for the flow correction, ΔQ, which is to be applied to the assumed values of flow to balance the head losses of the circuit, is usually written with summation signs.

$$\Delta Q = - \frac{\Sigma kQ^n}{\Sigma n\ kQ^{n-1}}$$

It should be noted that ΣkQ^n is algebraic while $\Sigma n\ kQ^{n-1}$ is arithmetic. The value of ΔQ is not exact because the powers of ΔQ greater than unity have been neglected.

The Hardy Cross method consists of assuming values for the flows in the various pipes of the network such that the algebraic sum of the discharge toward any junction point is zero, computing the flow correction, ΔQ, for these assumed flows, applying the flow correction, ΔQ, to these assumed flows, and then using the resultant flows to recompute a new flow correction. This process is repeated until the flow correction becomes zero.

The Darcy-Weisbach equation states that the head loss varies with the second power of the discharge.

$$h = kQ^2$$

In this equation,

$$k = \frac{16f\ L}{2g\ \pi^2\ D^5}$$

in which,

f = Darcy-Weisbach resistance coefficient

L = length of the pipe

g = magnitude of gravitational attraction

π = ratio of the circumference of a circle to its diameter

D = inside diameter of the pipe

If the Darcy-Weisbach equation is used in determining ΔQ, the following results are obtained.

$$h_a - h_b = 0$$

$$h_a = k_a\ (Q_a + \Delta Q)^2$$

$$h_b = k_b\ (Q_b - \Delta Q)^2$$

$$k_a\ (Q_a + \Delta Q)^2 - k_b\ (Q_b - \Delta Q)^2 = 0$$

$$(k_a Q_a^2 - k_b Q_b^2) + (2\ k_a Q_a + 2 k_b Q_b)\ \Delta Q + (k_a - k_b)(\Delta Q)^2 = 0$$

$$\Sigma k\ Q^2 + 2(\Sigma kQ)\ \Delta Q + (\Sigma k)(\Delta Q)^2 = 0 \qquad (1)$$

It should be noted that $\Sigma k\ Q^2$ and Σk are algebraic while $\Sigma k\ Q$ is arithmetic. If powers of ΔQ greater than unity are neglected the Hardy Cross flow correction can be obtained by solving equation (1) for ΔQ.

$$\Delta Q = -\frac{\Sigma k\ Q^2}{2\ \Sigma\ k\ Q}$$

If the Σk in equation (1) is zero then the Hardy Cross flow correction is the exact flow correction.

An exact solution for ΔQ can be made by solving equation (1) for ΔQ using the quadratic formula.

$$\Delta Q = \frac{-\Sigma kQ \pm \sqrt{(\Sigma kQ)^2 - (\Sigma k)(\Sigma k\,Q^2)}}{\Sigma k}$$

When a circuit is balanced the algebraic summation of the head losses around that circuit, $\Sigma k\,Q^2$, are zero, and ΔQ for that circuit is zero. Substituting the conditions for a balanced circuit in the exact formula for the flow correction,

$$0 = \frac{-\Sigma kQ \pm \sqrt{(\Sigma kQ)^2 - (\Sigma k)(0)}}{\Sigma k}$$

$$0 = \frac{-\Sigma kQ \pm |\Sigma k\,Q|}{\Sigma k}$$

In order for ΔQ in the exact equation to be equal to zero when the circuit is balanced, the positive sign for the radical must be used. The negative sign on the radical is, therefore, an extraneous root and the exact equation can be written,

$$\Delta Q = \frac{-\Sigma kQ + \sqrt{(\Sigma kQ)^2 - (\Sigma k)(\Sigma kQ^2)}}{\Sigma k}$$

Rapidity of convergence of the Hardy Cross method.—The rapidity of convergence of the Hardy Cross method depends upon the accuracy of the Hardy Cross flow correction and, when the network consists of more than one circuit, upon the effect of the flow corrections which are carried over from other circuits when a single pipe is simultaneously a member of more than one circuit.

The accuracy of the Hardy Cross flow correction, ΔQ_{HC}, may be determined by comparing it to the exact flow correction, ΔQ_{EX}. For purposes of simplification let,

$$\Sigma k = a$$

$$\Sigma k\,Q = b$$

$$\Sigma k\,Q^2 = c$$

$$\Delta Q_{HC} = -\frac{c}{2b}$$

$$\Delta Q_{EX} = \frac{-b + \sqrt{b^2 - ac}}{a}$$

In order to simplify the exact formula, the radical, $\sqrt{b^2 - ac}$, is expanded by the binominal theorem. To achieve a convergent series two cases must be considered.

Case I, ($|b^2| > |ac|$).

$$(b^2 - ac)^{1/2} = b - \frac{ac}{2b} - \frac{a^2c^2}{8\,b^3} - \frac{a^3c^3}{16\,b^5} \ldots$$

$$\Delta Q_{EX} = -\frac{c}{2b} - \frac{ac^2}{8\,b^3} - \frac{a^2\,c^3}{16\,b^5} \ldots$$

$$\frac{\Delta Q_{EX}}{\Delta Q_{HC}} = 1 + \frac{8c}{4\,b^2} + \frac{a^2c^2}{8\,b^4} \ldots$$

$$\frac{\Delta Q_{EX}}{\Delta Q_{HC}} = 1 + \frac{(\Sigma k)(\Sigma kQ^2)}{4(\Sigma kQ)^2} \quad \frac{(\Sigma k)^2\,(\Sigma kQ^2)^2}{8(\Sigma kQ)^4} \ldots$$

Since the series is converging and ΣkQ is always positive, the following conclusions can be made for Case I:

1. When both Σk and ΣkQ^2 are positive or negative the Hardy Cross flow correction will be too small.
2. When Σk and ΣkQ^2 are of opposite signs the Hardy Cross flow correction is too large.
3. The Hardy Cross flow correction becomes less accurate as ΣkQ decreases in relation to the product of Σk and ΣkQ^2.

Case II, ($|ac| > |b^2|$). It is possible for either of both a or c to have negative signs. b is always positive. In Case II for the expansion of the radical, $\sqrt{b^2 - ac}$, in which $|ac| > |b^2|$, it is necessary for a and c to have opposite signs otherwise the solution is an imaginary number and the case has no physical significance.

$$(-ac + b^2)^{1/2} = +(-ac)^{1/2} + \frac{b^2}{2(-ac)^{1/2}} - \frac{b^4}{8(-ac)^{3/2}}$$

$$+ \frac{b^6}{16(-ac)^{5/2}} - \frac{5b^8}{128(-ac)^{7/2}} \cdots$$

$$\Delta Q_{EX} = -\frac{b}{a} + \frac{(-ac)^{1/2}}{a} + \frac{b^2}{2a\,(-ac)^{1/2}} - \frac{b^4}{8a(-ac)^{3/2}}$$

$$+ \frac{b^6}{16a(-ac)^{5/2}} - \frac{5b^8}{128a(-ac)^{7/2}} \cdots$$

$$\frac{\Delta Q_{EX}}{\Delta Q_{HC}} = + \frac{2b^2}{ac} - \frac{2b(-ac)^{1/2}}{ac} - \frac{b^3}{ac(-ac)^{1/2}} + \frac{b^5}{4ac(-ac)^{3/2}}$$

$$- \frac{b^7}{8ac(-ac)^{5/2}} + \frac{5b^9}{64ac(-ac)^{7/2}} \cdot \cdot \cdot$$

Since in order for Case II to be real and have meaning, a and c must be of opposite signs, the expression for $\frac{\Delta Q_{EX}}{\Delta Q_{HC}}$ can be written in absolute values as follows:

$$\frac{\Delta Q_{EX}}{\Delta Q_{HC}} = - \left|\frac{2b^2}{ac}\right| + \left|\frac{2b}{(ac)^{1/2}}\right| + \left|\frac{b^3}{(ac)^{3/2}}\right| - \left|\frac{b^5}{4(ac)^{5/2}}\right|$$

$$+ \left|\frac{b^7}{8(ac)^{7/2}}\right| - \left|\frac{5b^9}{64(ac)^{9/2}}\right| \cdot \cdot \cdot$$

Let $R = \left|\frac{b^2}{ac}\right| = \left|\frac{(\Sigma kQ)^2}{(\Sigma k)(\Sigma kQ^2)}\right|$

Since in Case II, $|ac| > |b^2|$, R is always less than unity.

$$\frac{\Delta Q_{EX}}{\Delta Q_{HC}} = - 2R + 2R^{1/2} + R^{3/2} - \frac{R^{5/2}}{4} + \frac{R^{7/2}}{8} - \frac{5R^{9/2}}{64} \cdot \cdot \cdot$$

The following conclusions can be drawn in Case II:

1. Since R is always less than unity the summation of the series is always less than unity and the Hardy Cross flow correction is always greater than the exact flow correction.

2. The error in the Hardy Cross flow correction increases as the value of R decreases.

The Hardy Cross flow correction is equal to the exact flow correction when the Σk equals zero as was explained in the derivation of the exact formula.

The rapidity of convergence in a single circuit depends only upon the accuracy of the Hardy Cross flow correction. When a network consists of more than one circuit some pipes appear simultaneously in two circuits and therefore must have two flow corrections applied to them. Either one of these flow corrections, if computed by the exact formula, would cause the head losses of the circuit being considered to balance, but when both flow corrections are applied simultaneously to the pipe, the head losses around the circuit will not balance. Thus, the rapidity of convergence of the Hardy Cross method differs with each individual network. Convergence is more rapid when the k values of the pipes of the network are of the same magnitude, and convergence becomes less rapid as the difference in magnitude of the k values increases. Convergence also depends upon the accuracy of the first assumptions for the flow in the pipes.

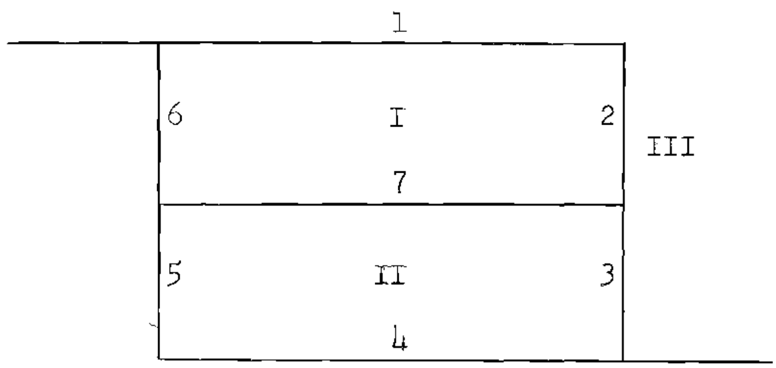

Fig. 3. A Simple Pipe Network Consisting of More Than One Circuit.

At times convergence will be more rapid if an extra circuit is considered such that each pipe will appear in two circuits. In Fig. 3, circuit I consists of pipes 1, 2, 6 and 7, and circuit II consists of pipes 3, 4, 5 and 7. These two circuits are the only circuits that need be considered to work the problem. However, if circuit III, which consists of pipes 1, 2, 3, 4, 5 and 6 is considered, each pipe will appear in two circuits and will therefore have two flow corrections to be applied. If any flow correction is large then an average value of these flow corrections should be used. If all flow corrections are small then both flow corrections should be applied to each pipe.

As a problem is solved, the successive values of the calculated flow in some pipes oscillate around the correct value of flow while in other pipes the successive values of the calculated flow increase or decrease asymptotically. Convergence is more rapid when the successive values of flow are oscillating. It is, therefore, wise to make a new assumption for the values of flows in pipes whose flows are increasing or decreasing asymptotically based on the trend that has been established for flow in these pipes.

CHAPTER IV

ANALYSIS BY THE HARDY CROSS METHOD OF THE FLOW DISTRIBUTION IN A PIPE NETWORK CONTAINING ELEVATED RESERVOIRS

The Derivation of the Hardy Cross Method for Networks Containing Elevated Reservoirs

The analysis of the flow distribution in a pipe network containing elevated reservoirs by the Hardy Cross method (9) is based on the inclusion of the potential heads of the elevated reservoirs in the Hardy Cross formula for the flow correction.

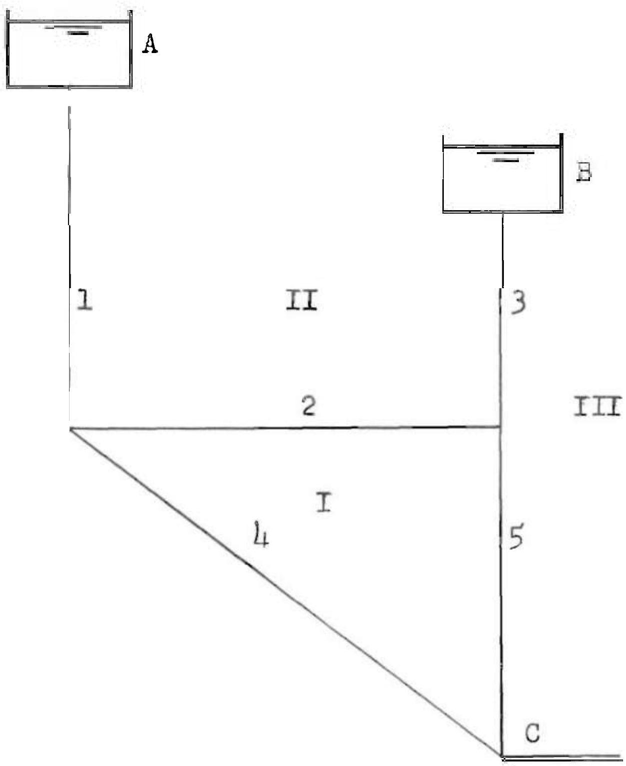

Fig. 4. A Simple Network Containing Elevated Reservoirs

Fig. 4 is a line diagram of a simple network which contains elevated reservoirs "A" and "B" which have potential heads, h_a and h_b, respectively. The solution of this problem is governed by the three general laws which afford the solution of all flow distribution problems.

1. The algebraic sum of the flows toward any junction point is zero.
2. In a closed circuit containing only pipes the algebraic sum of the head losses around the circuit is zero. In an open circuit containing reservoirs and the connecting pipes, the algebriac sum of the head losses around the circuit is equal to the difference in potential heads of the reservoirs.
3. The head loss varies with some power of the discharge.

Circuit II of Fig. 4 consists of reservoir "A," pipes 1, 2 and 3 and reservoir "B." For this circuit when the proper distribution of flow is achieved the algebraic sum of the head losses in the pipes must equal the difference in potential heads of the reservoirs, or the algebraic sum of the head losses minus the algebraic sum of the potential heads must equal zero. If clockwise flow is arbitrarily considered positive and counterclockwise flow is arbitrarily considered negative and the potential heads are arbitrarily considered positive or negative depending upon whether they tend to cause clockwise or counterclockwise flow, respectively, the following equation can be written:

$$-k_1(Q_1 - \Delta Q)^n - k_2(Q_2 - \Delta Q)^n + k_3(Q_3 + \Delta Q)^n - (-h_a + h_b) = 0$$

in which,

h_a and h_b = potential heads

Q_1, Q_2, Q_3 = assumed values of flow

$Q \pm \Delta Q$ = correct value of flow

ΔQ = flow correction to be applied to the assumed values of flow to achieve the correct values of flow.

If the terms raised to the power, n, are expanded by the binomial series and if powers of ΔQ greater than unity are neglected,

$$-k_1Q_1^n - k_2Q_2^n + k_3Q_3^n + nk_1Q_1^{n-1}\Delta Q + nk_2Q_2^{n-1}\Delta Q + nk_3Q_3^{n-1}\Delta Q$$

$$-(-h_a + h_b) = 0$$

$$\Sigma kQ^n + (\Sigma nkQ^{n-1})\Delta Q - \Sigma h = 0$$

$$\Delta Q = \frac{\Sigma h - \Sigma kQ^n}{\Sigma nkQ^{n-1}}$$

in which,

Σh = algebraic summation of the h terms

ΣkQ^n = algebraic summation of the kQ^n terms

ΣnkQ^{n-1} = arithmetic summation of the nkQ^{n-1} terms.

If the Darcy-Weisbach is used,

$$\Delta Q = \frac{\Sigma h - \Sigma kQ^2}{2\Sigma kQ}$$

In order to simplify the mechanics of the solution a sign convention can be arbitrarily adopted which will automatically result in the arithmetic and algebraic summations that are desired. The various solutions for ΔQ, where the Darcy-Weisbach equation is used, require that the Σk and ΣkQ^2 be algebraic and the ΣkQ be arithmetic. If clockwise flow and the k factor for clockwise flow are considered positive and counterclockwise flow and the k factor for counterclockwise flow are considered negative, the correct summations for terms and products of terms will result automatically, because the Σk and ΣkQ^2 will always be algebraic and ΣkQ will always be arithmetic.

In a circuit containing elevated reservoirs the algebraic summation of the head losses must be subtracted arithmetically from the algebraic summation of the potential heads. If the signs on the potential heads are reversed, that is if the potential heads which tend to cause counterclockwise flow are considered positive and the potential heads which tend to cause clockwise flow are considered negative, an algebraic summation of the potential heads and the head losses can be used.

In systems which contain elevated reservoirs two types of problems exist:

1. A constant flow is taken from the system.
2. Free flow is allowed from the system.

When constant flow is taken from the system in Fig. 4 two circuits must be considered: Circuit I which consists of pipes 2, 4 and 5 and circuit II which consists of reservoir "A," pipes

1, 2 and 3 and reservoir "B." If more rapid convergence is desired, Circuit III which consists of reservoir "B," pipes 3, 5, 4 and 1 and reservoir "A" must be considered also. Flows are assumed for the various pipes such that the flow toward any junction point is zero. The problem is then solved by the successive steps of the Hardy Cross method.

If free flow is allowed from the system in Fig. 4, the head at point "C" must be known. Point "C" is then considered as another reservoir having a known potential head. Circuit I consisting of pipes 2, 4 and 5, circuit II consisting of reservoirs "A" and "B" and pipes 1, 2 and 3, and circuit III consisting of reservoir "B," pipes 3 and 5, and point "C" must be considered. If more rapid convergence is desired, circuit IV consisting of point "C," pipes 4 and 1, and reservoir "A" must be considered also. The problem is then solved by the successive steps of the Hardy Cross method.

Analyses of Systems With Elevated Reservoirs for Cases of Constant and Free Flow From the System.

Studies of systems containing elevated reservoirs while the reservoirs are being allowed to empty are presented for both constant and free flow from the system. Each study contains an illustrative solution for the analysis of the flow distribution in the network by the Hardy Cross method.

Problem I, Constant Flow from a System Containing Elevated Reservoirs.-- Fig. 5 is a line diagram of the system showing the elevations and areas of the reservoirs and the lengths of the pipes. All pipe

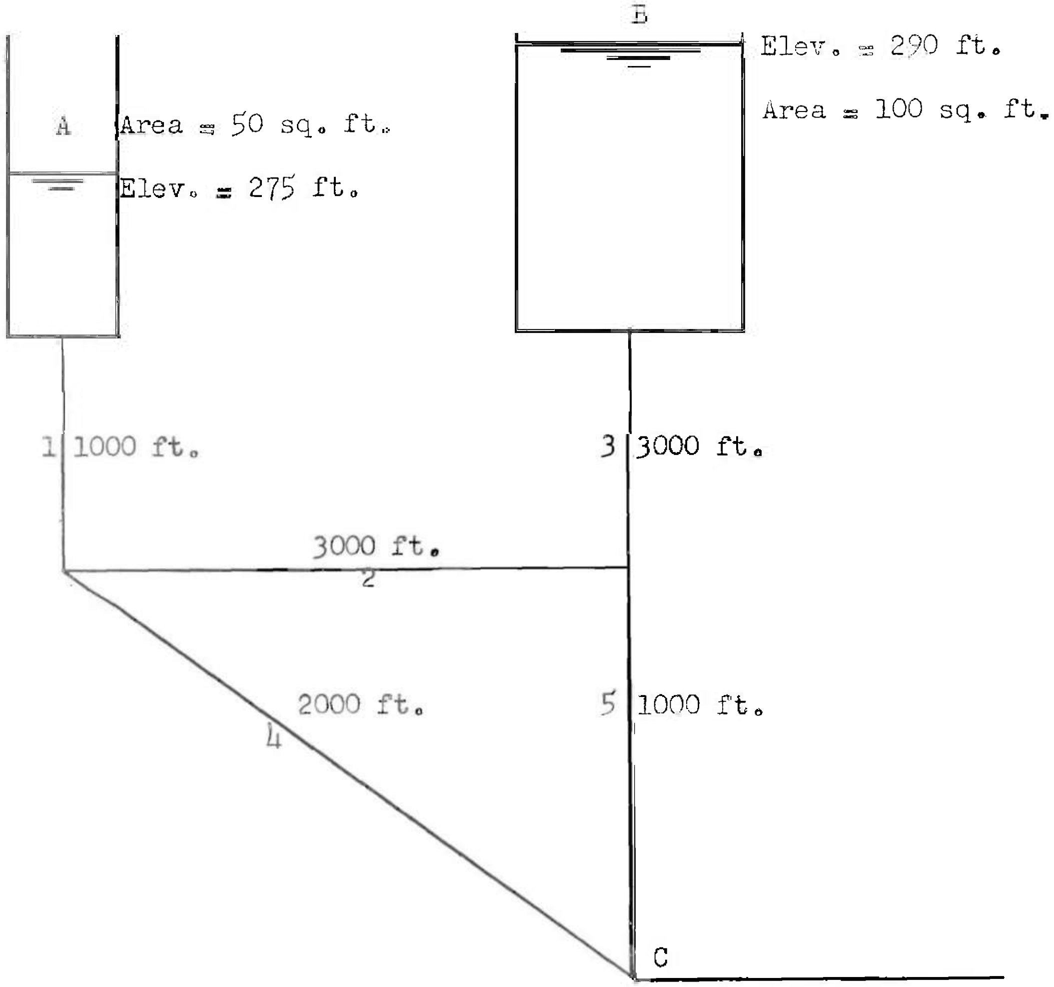

Fig. 5. Line Diagram of Network in Problems I and II.

is galvanized iron pipe of eight inch diameter. A constant flow of two cubic feet per second is drawn from the system. The Darcy-Weisbach resistance coefficient is assumed to be 0.020. The Darcy-Weisbach equation is used.

$$h = kQ^2$$

in which,

$$k = \frac{16f\ L}{2g\ \pi^2\ D^5}$$

in which,

f = Darcy-Weisbach resistance coefficient

L = length of pipe

D = inside diameter of the pipe

The k factors for the various pipes can be computed from the above formula. The k factor for pipe 1 is found as follows:

$$k_1 = \frac{16 f_1 L_1}{2g \pi^2 D_1^5} = \frac{16(0.020)(1000)}{2(32.2)(3.14)^2(0.67)^5} = 3.82 \text{ sec.}^2 \text{ per ft.}^5$$

The k factors for pipes 2, 3, 4 and 5 are computed in a similar manner and are found to be 11.46, 11.46, 7.64 and 3.82, respectively.

Tabular Solution for Flow Distribution of
Problem I, Initial Condition

Step 1.

Pipe	k	Qcfs	kQ^2	2kQ	Q		Qcfs
A			+275.00				
1	-3.82	-0.70	-1.87	+5.35	-0.07	+0.02	-0.75
2	+11.46	+0.30	+1.03	+6.88	-0.07	-0.16	+0.07
3	+11.46	+1.30	+19.40	+29.80	-0.07	+0.02	+1.25
B			-290.00				
			Σ +3.16	+42.03			
			ΔQ = -0.07				
A			-275.00				
1			+1.87	+5.35	-0.02		
4	+7.64	+1.00	+7.64	+15.28	-0.02	-0.16	+0.82
5	-3.82	-1.00	-3.82	+7.64	-0.02	-0.16	-1.18
3			-19.40	+29.80	-0.02		
B			+290.00				
			Σ +1.29	+58.07			
			ΔQ = -0.02				
4			-7.64	+15.28	+0.16		
5			+3.82	+7.64	+0.16		
2			-1.03	+6.88	+0.16		
			Σ -4.85	+29.80			
			ΔQ = +0.16				

Tabular Solution for Flow Distribution of
Problem I, Initial Condition

Step 2.

Pipe	k	Qcfs	kQ^2	2kQ	Q		Qcfs
A			+275.00				
1	-3.82	-0.75	-2.15	+5.73	-0.02	-0.02	-0.79
2	+11.46	+0.07	+0.06	+1.61	-0.02	+0.01	+0.06
3	+11.46	+1.25	+17.95	+28.65	-0.02	-0.02	+1.21
B			-290.00				
			Σ +0.86	+35.99			
			ΔQ = -0.02				
A			-275.00				
1			+2.15	+5.73	+0.02		
4	+7.64	+0.82	+5.14	+12.53	+0.02	+0.01	+0.85
5	-3.82	-1.18	-5.32	+9.02	+0.02	+0.01	+1.15
3			-17.95	+28.65	+0.02		
B			+290.00				
			Σ -0.98	+55.93			
			ΔQ = +0.02				
4			-5.14	+12.53	-0.01		
5			+5.32	+9.02	-0.01		
2			-0.06	+1.61	-0.01		
			Σ +0.12	+23.16			
			ΔQ = -0.01				

A study of the system while the reservoirs were being allowed to empty was made by computing the new elevations of the water level in the reservoirs at specified time intervals and then recomputing the distribution of flow for the new potential heads of the reservoirs. Calculations of the flow were made at five hundred second intervals. Results of the study are shown graphically in Fig. 6 where the flow from reservoir "A," Q_A, the flow from reservoir "B," Q_B and the difference in elevation of the water level of the two tanks, H_{AB}, are plotted against time.

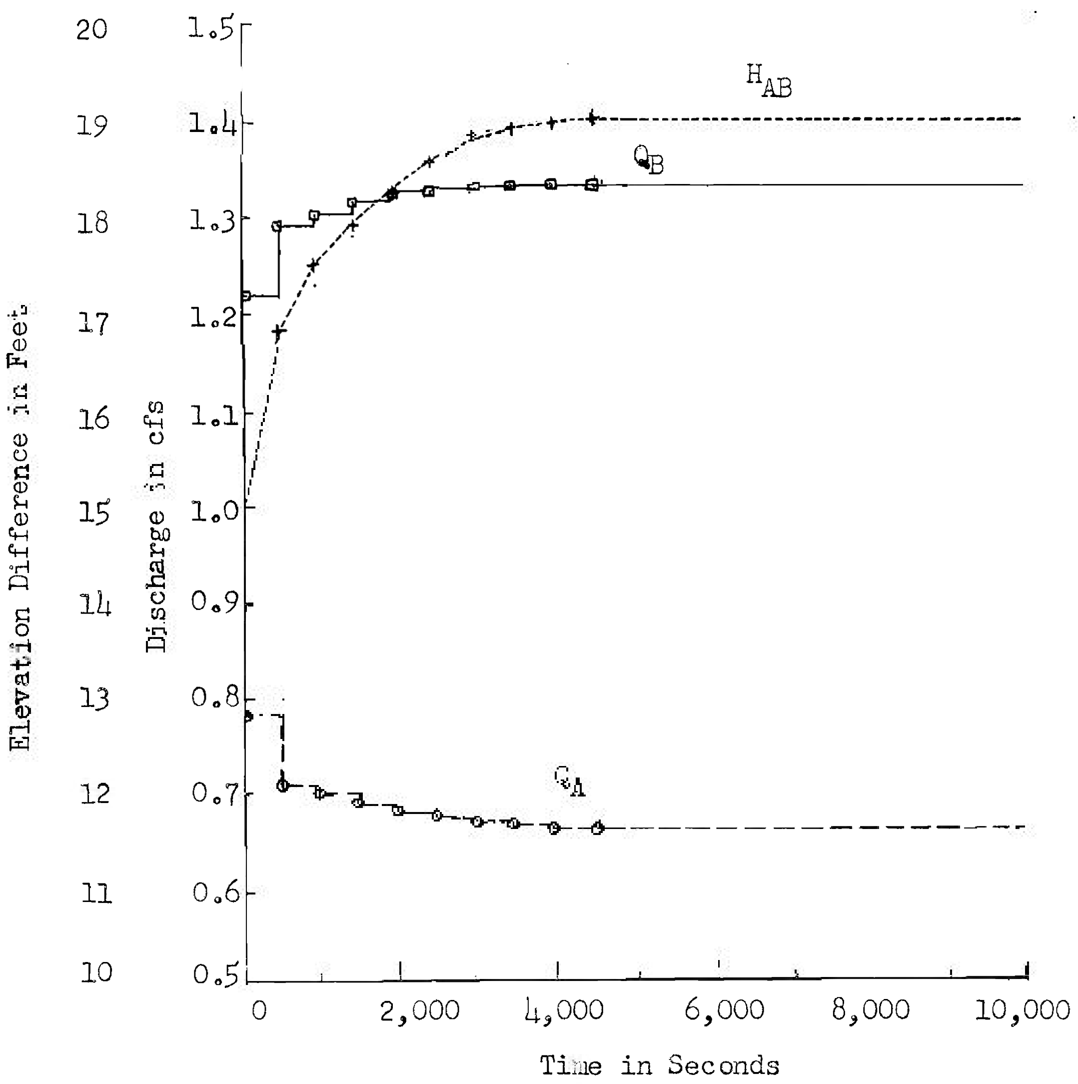

Fig. 6. Solution of Problem I

An examination of Fig. 6 shows that the flows from the reservoirs and the difference in elevation of the water levels of the two reservoirs tend to become constant as time progresses. These facts are true of all systems which contain elevated prismatic reservoirs when a constant flow is taken from the system.

In Fig. 4, which is a line diagram of a system containing elevated prismatic reservoirs, if a constant total flow, Q, is taken from the system the sum of the flows from reservoir "A" and reservoir "B" must be constant. The rates of flows from the reservoirs depend upon the resistance of the system and upon the potential heads of the reservoirs which in turn depend upon the rate of flow from the reservoirs and the area of the reservoirs. In an initial condition, the initial rate of flow from reservoir "A" will have some value, Q_A and the initial rate of flow from reservoir "B" will have some value Q_B. Over an increment of time, t, reservoir "A" will fall and increment of elevation, $\frac{Q_A t}{A_A}$, and reservoir "B" will fall an increment of elevation, $\frac{Q_B t}{A_B}$, where A_A and A_B are the areas of reservoirs "A" and "B," respectively. If these increments of fall in elevation of the reservoirs are equal, the flows from the reservoirs will remain constant and the difference in elevation of the reservoirs will remain constant. However, if the increment of fall of reservoir "A" is greater than the increment of fall of reservoir "B," then, at the start of the next interval of time the difference in elevation of the reservoirs will have decreased causing Q_A to decrease by some value, ΔQ, and Q_B to increase by ΔQ since the total flow is constant. Thus, during the second increment of time reservoir "A" will fall $\frac{(Q_A - \Delta Q)t}{A_A}$ and reservoir "B" will fall $\frac{(Q_B + \Delta Q)t}{A_B}$. In the first increment of time the increment of fall of reservoir "A" was greater than the increment of fall of reservoir "B," however, in the second interval of time the increment of fall of reservoir "A" decreases and the increment of fall of reservoir "B" increases. Thus, with the progression of time the increments of fall of the reservoirs tend to become equal. If the increments of fall of the reservoirs are equal over a period of time, then the ratio of the flow from any reservoir to the area of that reservoir is equal to the ratio of the flow from any other reservoir to the area of the other reservoir. Thus, the final flow from each reservoir in a system, from which a constant flow is being taken and which contains elevated prismatic reservoirs, can be determined by solving simultaneously, the equations which can be obtained from the equality of the individual ratios of the flow from the reservoirs to the area of the reservoirs, and the equation that states that the sum of the individual flows from the reservoirs is equal to the total flow out of the system, Q.

$$\frac{Q_A}{A_A} = \frac{Q_B}{A_B} = \frac{Q_C}{A_C}$$

$$Q_A + Q_B + Q_C = Q$$

$$Q_A + \frac{A_B}{A_A} Q_A + \frac{A_C}{A_A} Q_A = Q$$

$$Q_A = \frac{Q}{1 + \frac{A_B}{A_A} + \frac{A_C}{A_A}}$$

Problem II. Free Flow From a System Containing Elevated Reservoirs.--The system that is shown in Fig. 5 and was used in problem I is used for problem II. In problem II the potential head of point "C" is zero and free flow is allowed from the system at this point. In the first step of the tabular solution an average value of ΔQ is used for the flow correction.

Tabular Solution for Flow Distribution of
Problem II, Initial Condition

Step 1.

Pipe	k	Qcfs	kQ^2	2kQ		Q	Qcfs
A			+275.00				
1	-3.82	-5.50	-115.7	+42	-0.16	-0.36	-5.76
2	-11.46	-1.50	-25.8	+34	-0.16	+0.14	-1.51
3	+11.46	+4.00	+183.3	+92	-0.16	-0.06	+3.89
B			-290.0				
			Σ +26.8	+168			
			ΔQ = -0.16				
B			+290.0				
3			-183.3	+92	+0.06		
5	-3.82	-5.50	-115.7	+42	+0.06	+0.14	-5.40
C			0.0				
			Σ -9.0	+136			
			ΔQ = +0.06				
C			0.0				
4	+7.64	+4.00	+122.3	+61	+0.36	+0.14	+4.25
1			+115.7	+42	+0.36		
A			-275.0				
			Σ -37.0	+103			
			ΔQ = +0.36				
4			-122.3	+61	-0.14		
5			+115.7	+42	-0.14		
2			+25.8	+34	-0.14		
			Σ +19.2	+137			
			ΔQ = -0.14				

Step 2.

A			+275.0				
1	-3.82	-5.76	-126.9	+44	-0.03	-0.09	-5.88
2	-11.46	-1.51	-26.1	+35	-0.03	0.00	-1.54
3	+11.46	+3.89	+173.3	+85	-0.03	+0.04	+3.90
B			-290.0				
			Σ +5.3	+164			
			ΔQ = -0.03				
B			+290.0				
3			-173.3	+85	-0.04		
5	-3.82	-5.40	-111.4	+41	-0.04	0.00	-5.44
C			0.0				
			Σ +5.3	+126			
			ΔQ = -0.04				

(Continued)

Tabular Solution for Flow Distribution of
Problem II, Initial Condition

Step 2. (Continued)

Pipe	k	Qcfs	kQ^2	2kQ		Q	Qcfs
C			0.0				
4	+7.64	+4.25	+138.0	+65	+0.09	0.00	+ 4.34
1			+126.9	+44	+0.09		
A			-275.0				
			Σ -10.1	+109			
			ΔQ = +0.09				
4			-138.0	+65	0.00		
5			+111.4	+41	0.00		
2			+26.1	+35	0.00		
			Σ -0.5	+141			
			ΔQ = 0.00				

A study of the system while the reservoirs were being allowed to empty was made by computing the new elevations of the water level in the reservoirs at specified time intervals and then recomputing the distribution of the flow for the new potential heads of the reservoirs. Calculations of the flow were made at 500 second intervals until the reservoirs were empty. Results of the study are shown graphically in Fig. 7 where the flow from reservoir "A," Q_A, the flow from reservoir "B," Q_B, and the difference in elevation of the water level of the two reservoirs, H_{AB}, are plotted against time.

An examination of Fig. 7 shows that the flows from the reservoirs and the difference in elevation of the reservoirs are dependent upon the individual network which is being considered.

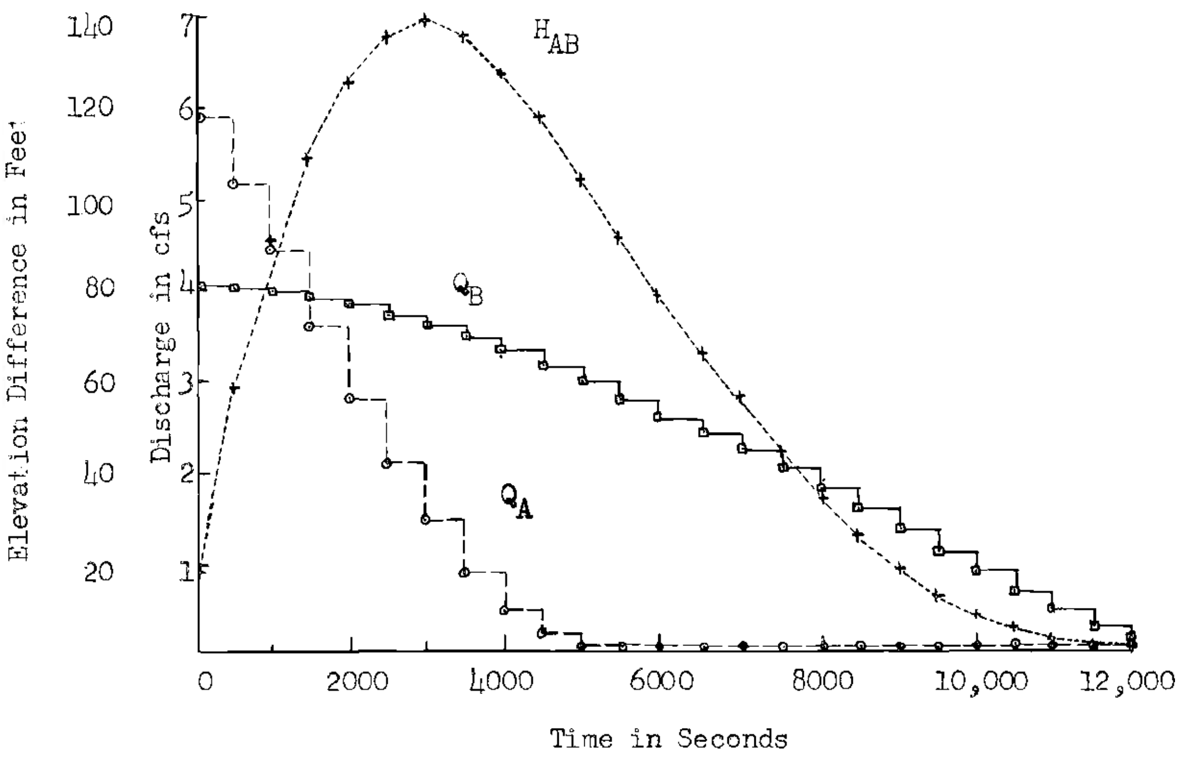

Fig. 7. Solution of Problem II

CHAPTER V

CONCLUSIONS

The conclusions concerning the use of the various methods of analyzing the distribution of flow in pipe networks containing elevated reservoirs are as follows:

1. If the network is simple and if the distribution of flow is required for only a few conditions, the Hardy Cross method is advisable.
2. If the network is complicated and if the distribution of flow is required for many conditions, the electric analogy is advisable.

The conclusions concerning the accuracy of the Hardy Cross flow correction are as follows:

1. The Hardy Cross flow correction is exact when the summation of the k factors is equal to zero.
2. The Hardy Cross flow correction is too small in any one circuit when both the summation of the k factors and the summation of the head losses are of the same sign.
3. The Hardy Cross flow corrections are too large in all of the three other cases.

The conclusions concerning the rapidity of convergence of the Hardy Cross method are as follows:

1. Convergence is more rapid when the Hardy Cross flow corrections are accurate, when the assumed values of flows are accurate, and when the k factors are of relatively the same magnitude.
2. In the solution of a problem more rapid convergence can be achieved if an extra circuit is considered such that each pipe will have two flow corrections to be applied. If any flow correction is lagre then an average value should be applied. If all flow corrections are small then both should be applied.

The conclusions concerning the flows in a system containing elevated reservoirs are as follows:

1. The flows in a system containing elevated reservoirs can be computed by the Hardy Cross method.
2. If a constant flow is taken from the system while the reservoirs are emptying the flows in the system and the difference in elevation of the water levels in the reservoirs will eventually reach a steady state in which the flows from the reservoirs are proportional to the areas of the reservoirs providing the system is undisturbed, the reservoirs are prismatic, and none of the reservoirs empty before this steady state is reached.
3. If free flow is allowed from the system while the reservoirs are emptying the flows in the system and the difference in elevation of the water levels in the reservoirs will depend upon the individual network being considered.

BIBLIOGRAPHY

(1) Freeman, J. R., "Graphical Solution of Water Distribution Problems," Journal of the New England Water Works Association, 7, 1892, p. 49.

(2) Howland, W. E., "Expansion of the Freeman Method For The Solution of Pipe Flow Problems," Journal of the New England Water Works Association, 48, 1934, p. 408.

(3) Aldrich, E. H., "Solution of Transmission Problems of a Water System," Transactions of the American Society of Civil Engineers, 103, 1938, p. 1579.

(4) Cross, H, "Analysis of Flow in Networks of Conduits or Conductors," University of Illinois Engineering Experiment Station Bulletin No. 286, November, 1936.

(5) Camp, T. R., "Hydraulics of Distribution Systems-Some Recent Developments in Methods of Analysis," Journal of New England Water Works Association, 57, 1943, p. 334.

(6) Camp, T. R., and Hazen, H. L., "Hydraulic Analysis by Electric Analyzer," Journal of the New England Water Works Association, 48, 1934, p. 383.

(7) McIlroy, M. S., Studies of Fluid Network Analysis and The Basic Design of an Electric Analyzer for Fluid Distribution Systems, Doctor's Thesis, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1946.

(8) McIlroy, M. S., "Water Distribution Systems Studied By a Complete Electrical Analogy," Journal of the New England Water Works Association, 65, 1951, p. 299.

(9) O'Connor, T. F., "A Rapid Method of Analyzing Flow in Water Distribution Systems Based on the Theory Developed by Professor Hardy Cross," Proceedings of the Maryland Delaware Water and Sewage Association, 1943.